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A SIMPLIFIED VARIATION OF PARAMETERS SOLUTION FOR THE MOTION OF AN ARBITRARILY-TORQUED ASYMMETRIC RIGID BODY

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A SIMPLIFIED VARIATION OF PARAMETERS SOLUTION FOR THE MOTION OF AN ARBITRARILY-TORQUED ASYMMETRIC RIGID BODY

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A normalized form of Euler's equations is rewritten using a variation of parameters approach with amplitudes and angular displacement as parameters. This new form is compact and yields a much more accurate numerically integrated solution over longer simulation intervals than a conventional integration of Euler's equations. The complete variation of parameters formulation involves the classical Jacobian elliptic functions as well as standard elliptic integrals. Because this formulation is developed in the fixed reference frame of the body's principal axes, these variation of parameters equations can be simply grouped with the dynamical equations for rotation, i.e. the variation of the Euler-Rodrigues parameters, to provide singularity free information about the attitude of the body in a local inertial reference frame. This formulation is then compared to the traditional formulation in the normalized body-frame to investigate error propagation behavior in computer simulations.

INTRODUCTION

Variational methods have played an important roll in solving ordinary differential equations since their introduction by John Bernoulli in the late seventeenth century. Perhaps the most famous application of a variational method, the variation of orbital elements or the variation of constants, was performed by Leonhard Euler between 1748 and 1752 to describe the mutual perturbations of Jupiter and Saturn. However, it was not until 1782 that Joseph-Louis Lagrange fully and completely developed the method of the variation of parameters (VOP) in an application involving cometary motion. His approach is widely used today particularly in Astrodynamics applications where perturbed two-body motion is considered.

Because the torque-free rigid body motion equations developed by Euler in 1754 admit an analytical solution, it seemed desirable to apply Lagrange's method of the variation of parameters to these equations. It was hoped that the resulting differential equations would

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demonstrate the same numerical robustness and accuracy enhancement that characterizes the two-body motion VOP equations. Indeed, this has been found to be true through a careful selection of VOP system parameters.

A VOP approach to Euler's equations and to the general orientational motion of rigid bodies has been investigated several times since the mid-1970's. Early work by Kraige and Junkins [8] and Donaldson and Jezewski [5] used the body's kinetic energy and angular momentum as the primary parameters in a general VOP scheme. The resulting equations were algebraically complex. The later work of Kraige and Skaar [9] and Kraige [7] did introduce less complexity to the form of the variational equations. More recently, Bond [1] investigated a VOP scheme for rigid body motion that utilized the case of symmetric, torque-free motion as the analytical basis for generating the parameters. Taken altogether, it was felt that improvements could be made through a more judicious selection of parameters developed from the analytical solution to the general problem of torque-free rotation.

In this paper, the analytical solution to the classical Euler's equations for rigid body motion is redeveloped to simplify the process of obtaining an analytical solution, as compared to traditional methods, and generate parameters that feel more natural and intrinsic to the problem. These parameters are used to develop VOP equations that are not algebraically complex, making them desirable for use in any numerical simulation. This approach is then applied to several constant torque problems to validate the claims of improved accuracy integrations over longer simulation periods. The benefit of improved accuracy is independent of the numerical integration scheme chosen to propagate the equations.

BODY FRAME FORMULATION

The usual method of developing an analytical solution for Euler's torque-free motion equations is cumbersome and tedious to follow [6]. Moreover, immediately invoking the conservation laws for energy and angular momentum to obtain the first integrals of motion in the body-frame complicates the solution development, since the resulting expressions fail to provide the easily recognized special function identities that can simplify the solution process. The following addresses these issues.

Normalizing Euler's Equations

The classical Euler's equations for rigid body motion are given by the coupled first-order system

$$A\dot{\omega}_1 = (B - C)\,\omega_2\omega_3 + M_1, \tag{1a}$$

$$B\dot{\omega}_2 = (C - A)\,\omega_1\omega_3 + M_2, \tag{1b}$$

$$C\dot{\omega}_3 = (A - B)\,\omega_1\omega_2 + M_3. \tag{1c}$$

The ω_i are the components of the angular velocity of the body expressed in a body-fixed principal-axis coordinate frame. The M_i are the body-fixed external torque components. A, B, and C are the principal-axis moments of inertia which are ordered, without loss of generality, such that $A < B < C$.

Rewriting Eqs. (1), we have

$$\dot{\omega}_1 = -D_1\omega_2\omega_3 + \frac{M_1}{A}, \tag{2a}$$

$$\dot{\omega}_2 = \quad D_2\omega_1\omega_3 + \frac{M_2}{B}, \tag{2b}$$

$$\dot{\omega}_3 = -D_3\omega_1\omega_2 + \frac{M_3}{C}, \tag{2c}$$

where

$$D_1 = (C - B)/A, \tag{3a}$$

$$D_2 = (C - A)/B, \tag{3b}$$

$$D_3 = (B - A)/C, \tag{3c}$$

with $D_i > 0$.

Upon performing a change of variables defined by the relations[1]

$$\Omega_i = \frac{\omega_i}{\sqrt{D_i}}, \tag{4a}$$

$$\frac{d\tau}{dt} = \sqrt{D_1 D_2 D_3}, \tag{4b}$$

the normalized form of the full Euler equations becomes

$$\Omega_1{}' = -\Omega_2\Omega_3 + G_1, \tag{5a}$$

$$\Omega_2{}' = \quad \Omega_1\Omega_3 + G_2, \tag{5b}$$

$$\Omega_3{}' = -\Omega_1\Omega_2 + G_3, \tag{5c}$$

where the prime indicates $d/d\tau$ and G_i are the rescaled external moments given by

$$G_1 = \frac{M_1}{AD_1\sqrt{D_2 D_3}}, \tag{6a}$$

$$G_2 = \frac{M_2}{BD_2\sqrt{D_1 D_3}}, \tag{6b}$$

$$G_3 = \frac{M_3}{CD_3\sqrt{D_1 D_2}}. \tag{6c}$$

Finally, taking $G_i = 0$ in Eqs. (5) we have the transformed torque-free Euler equations,

$$\Omega_1{}' = -\Omega_2\Omega_3, \tag{7a}$$

$$\Omega_2{}' = \quad \Omega_1\Omega_3, \tag{7b}$$

$$\Omega_3{}' = -\Omega_1\Omega_2. \tag{7c}$$

[1] The following transformation was developed simultaneously and independently of Livneh and Wei [10] for a different application. Also, see Richardson and Mitchell [11].

Two independent integrals of motion, c_1 and c_2, of the unperturbed Eqs. (7) are immediately obtained from the expressions

$$\Omega_1 {\Omega_1}' = \frac{d}{d\tau}\left(\frac{1}{2}{\Omega_1}^2\right) = -\Omega_1\Omega_2\Omega_3, \tag{8a}$$

$$\Omega_2 {\Omega_2}' = \frac{d}{d\tau}\left(\frac{1}{2}{\Omega_2}^2\right) = \quad \Omega_1\Omega_2\Omega_3, \tag{8b}$$

$$\Omega_3 {\Omega_3}' = \frac{d}{d\tau}\left(\frac{1}{2}{\Omega_3}^2\right) = -\Omega_1\Omega_2\Omega_3, \tag{8c}$$

and by inspection are seen to be

$${\Omega_1}^2 + {\Omega_2}^2 = {c_1}^2, \tag{9a}$$

$${\Omega_2}^2 + {\Omega_3}^2 = {c_2}^2. \tag{9b}$$

In the following theorems, these constants of integration are related to the traditional constants of angular momentum H and kinetic energy T.

Theorem 1. *The integrals of motion, ${c_1}^2 \equiv {c_1}^2\,(H,T)$ and ${c_2}^2 \equiv {c_2}^2\,(H,T)$ are given by*

$${c_1}^2 = \frac{2TC - H^2}{(C-A)(C-B)}, \tag{10a}$$

$${c_2}^2 = \frac{H^2 - 2TA}{(C-A)(B-A)}. \tag{10b}$$

Proof. Expressions for the angular momentum H and the total kinetic energy T are given by

$$2T = A{\omega_1}^2 + B{\omega_2}^2 + C{\omega_3}^2, \tag{11a}$$

$$H^2 = A^2{\omega_1}^2 + B^2{\omega_2}^2 + C^2{\omega_3}^2. \tag{11b}$$

Transforming $\omega_i \mapsto \Omega_i$ with Eqs. (3) and (9),

$$\begin{aligned}
2T &= AD_1{\Omega_1}^2 + BD_2{\Omega_2}^2 + CD_3{\Omega_3}^2 \\
&= \left({\Omega_1}^2 + {\Omega_2}^2\right) C - \left({\Omega_1}^2 - {\Omega_3}^2\right) B - \left({\Omega_2}^2 + {\Omega_3}^2\right) A \\
&= (C-B)\,{c_1}^2 + (B-A)\,{c_2}^2, \\
H^2 &= A^2D_1{\Omega_1}^2 + B^2D_2{\Omega_2}^2 + C^2D_3{\Omega_3}^2 \\
&= \left({\Omega_1}^2 - {\Omega_3}^2\right) AC - \left({\Omega_1}^2 + {\Omega_2}^2\right) AB + \left({\Omega_2}^2 + {\Omega_3}^2\right) BC \\
&= A\,(C-B)\,{c_1}^2 + C\,(B-A)\,{c_2}^2.
\end{aligned}$$

Forming $2TC - H^2$ and $H^2 - 2TA$,

$$\begin{aligned}
2TC - H^2 &= (B-A)\,(C-B)\,{c_1}^2, \\
H^2 - 2TA &= (C-A)\,(B-A)\,{c_2}^2,
\end{aligned}$$

and simplifying, we obtain the desired result. □

Theorem 2. *The integrals of motion $c_1{}^2$ and $c_2{}^2$ are non-negative.*

Proof. Forming $H^2 - 2TA$, for $\Omega_i \in \mathbb{R}$, using Eqs. (3), (4a), (11) and retaining the strict ordering of the principal moments of inertia, produces

$$H^2 - 2TA = \underbrace{(B-A)(C-A)\Omega_2{}^2}_{\geq 0} + \underbrace{(C-A)(B-A)\Omega_3{}^2}_{\geq 0}$$

$$= 2T\left(\frac{H^2}{2T} - A\right) \geq 0 \Rightarrow \frac{H^2}{2T} \geq A. \quad (12)$$

Likewise, forming $2TC - H^2$,

$$2TC - H^2 = \underbrace{(C-A)(C-B)\Omega_1{}^2}_{\geq 0} + \underbrace{(C-B)(C-A)\Omega_2{}^2}_{\geq 0}$$

$$= 2T\left(C - \frac{H^2}{2T}\right) \geq 0 \Rightarrow \frac{H^2}{2T} \leq C. \quad (13)$$

Combining the two expressions above,

$$A \leq \frac{H^2}{2T} \leq C. \quad (14)$$

From the problem specification, we have $C > B > A > 0$. Thus, invoking Theorem 1 completes the proof. □

Torque-Free Analytical Solution Revisited

A solution for the unperturbed system, Eqs. (7), is developed by first considering Eq. (7b). Combining Eq. (7b) with Eqs. (9) gives

$$\begin{aligned}\Omega_2{}' &= \sqrt{c_1{}^2 - \Omega_2{}^2}\sqrt{c_2{}^2 - \Omega_2{}^2}, \\ &= c_1 c_2 \sqrt{1 - \Omega_2{}^2/c_1{}^2}\sqrt{1 - \Omega_2{}^2/c_2{}^2}, \\ &\triangleq c_1 c_2 \sqrt{1 - s^2}\sqrt{1 - k^2 s^2},\end{aligned} \quad (15)$$

or in integral form

$$\int \frac{d\Omega_2}{\sqrt{c_1{}^2 - \Omega_2{}^2}\sqrt{c_2{}^2 - \Omega_2{}^2}} = \int d\tau. \quad (16)$$

Choosing $s = \Omega_2/c_1$, the modulus $k = c_1/c_2$, and restricting the modulus such that $0 < k < 1$, we recognize Eq. (16) from Byrd and Friedman [4] Eq. (BF120.01) as the inverse of a generalized sine function,

$$\text{sn}^{-1}(s) = c_2(\tau - \tau_0) \triangleq u, \quad (17)$$

where $\operatorname{sn} u$ is the Jacobian elliptic function *sine amplitude* u. Using this along with Eqs. (9) and (BF121.00) produces the familiar torque-free solution in terms of the Jacobian elliptic functions

$$\Omega_1 = c_1 \operatorname{cn} u, \tag{18a}$$
$$\Omega_2 = c_1 \operatorname{sn} u, \tag{18b}$$
$$\Omega_3 = c_2 \operatorname{dn} u, \tag{18c}$$
$$u = c_2(\tau - \tau_0), \tag{18d}$$

considering u to be the intrinsic angular argument.

PARAMETER SELECTION

From Eqs. (18), the quantities c_1, c_2 and τ_0 form a natural set of parameters for the characterization of the solution to the torque-free Euler equations. These quantities are analogous to a harmonic oscillator's amplitude, frequency, and initial time constant, respectively.

TYPES OF MOTION

The particular torque-free solution described by Eqs. (1) is typically specified by a relationship involving the angular momentum, total kinetic energy, and the intermediate moment of inertia[2], viz

$$\frac{H^2}{2T} \gtreqless B. \tag{19}$$

If $H^2/2T < B$, the motion is described as *epicycloidal*. In this case, the outside of the momentary body cone[3] appears to roll upon the outside of the momentary space cone[4]. If $H^2/2T > B$, the motion is described as *pericycloidal*, where the inside of the momentary body cone appears to roll upon the outside of the momentary space cone. In the transitional case of $H^2/2T = B$, the motion is degenerate and for the special case of $\omega_1 = \omega_3 = 0$, the angular momentum vector and the body angular velocity vector coincide. Lastly, the complete range of possible motion is delimited when we consider the upper and lower bounds given in Eq. (14). These bounds represent two additional degenerate motions from the perspective of the modulus.

From Rimrott [13], we have

$$k = \sqrt{\frac{H^2 - 2TC}{2TA - H^2} \cdot \frac{A - B}{B - C}}, \qquad \text{if } \frac{H^2}{2T} < B, \tag{FR5.11}$$

and

$$k = \sqrt{\frac{2TA - H^2}{H^2 - 2TC} \cdot \frac{B - C}{A - B}}, \qquad \text{if } \frac{H^2}{2T} > B. \tag{FR5.16}$$

[2]see Rimrott [13], pp. 105–111.

[3]A cone created by the precession of the body's angular velocity vector.

[4]A cone whose half angle lies between the angular momentum vector and the body angular velocity vector.

These two equations relate the type of motion to the structure of the modulus k. With this, it is possible to classify the types of motion.

Theorem 3. *The ratios of the parameters c_1 and c_2 that form the modulus k classify the motion described by the given variational equations as*

$$k = \frac{c_1}{c_2}, c_1 < c_2 \Leftrightarrow \frac{H^2}{2T} > B \quad \textit{(pericycloidal motion)}, \tag{20a}$$

$$k = \frac{c_2}{c_1}, c_2 < c_1 \Leftrightarrow \frac{H^2}{2T} < B \quad \textit{(epicycloidal motion)}, \tag{20b}$$

with the special degenerate cases

$$k = 0, c_1 = 0 \ \Leftrightarrow H^2 = 2TC \quad \textit{(harmonic)}, \tag{21a}$$

$$k = 0, c_2 = 0 \ \Leftrightarrow H^2 = 2TA \quad \textit{(harmonic)}, \tag{21b}$$

$$k = 1, c_1 = c_2 \Leftrightarrow H^2 = 2TB \quad \textit{(hyperbolic)}. \tag{21c}$$

Proof. This follows directly from Theorems 1, 2 and Eqs. (FR5.11), (FR5.16). □

VARIATION OF PARAMETERS

Obtaining a functional expression for τ_0 is difficult. Using either Lagrange's or Poisson's method to find the variation of τ_0 fails to provide a manageable solution due to the lack of sufficient elliptic function and integral identities necessary to render a suitable expression. As a result, it is more algebraically desirable to seek the variation of the argument u rather than τ_0 as in the following.

Variational Equations for $c_1 < c_2$

The variations of the two independent integrals of motion, c_1 and c_2, are obtained by direct differentiation of Eqs. (9) along with substitutions from Eqs. (5) and (18), yielding

$$c_1{}' = \frac{1}{c_1}(\Omega_1 G_1 + \Omega_2 G_2) = G_1 \operatorname{cn} u + G_2 \operatorname{sn} u, \tag{22a}$$

$$c_2{}' = \frac{1}{c_2}(\Omega_2 G_2 + \Omega_3 G_3) = kG_2 \operatorname{sn} u + G_3 \operatorname{dn} u. \tag{22b}$$

To develop a variational equation for u, we begin by writing the derivatives, with respect to τ, of the Eqs. (18a) and (18b) as

$$\Omega_1{}' = \frac{\partial \Omega_1}{\partial u}u' + \frac{\partial \Omega_1}{\partial c_1}c_1{}' + \frac{\partial \Omega_1}{\partial k}k', \tag{23a}$$

$$\Omega_2{}' = \frac{\partial \Omega_2}{\partial u}u' + \frac{\partial \Omega_2}{\partial c_1}c_1{}' + \frac{\partial \Omega_2}{\partial k}k'. \tag{23b}$$

Using the above to form ${\Omega_2}' \operatorname{cn} u - {\Omega_1}' \operatorname{sn} u$, we have

$$ {\Omega_2}' \operatorname{cn} u - {\Omega_1}' \operatorname{sn} u = [c_1 \operatorname{dn} u]\, u' - \frac{c_1 \operatorname{dn} u}{k {k_c}^2} \left[E(u) - k_c^2 u - k^2 \operatorname{sn} u \operatorname{cn} u / \operatorname{dn} u\right] k' \tag{24a} $$

where, ${k_c}^2 = 1 - k^2$ is the complementary modulus, $E(u)$ is the incomplete elliptic integral of the second kind, and k' is given by

$$ k' = \frac{1}{c_2} \left[G_1 \operatorname{cn} u + {k_c}^2 G_2 \operatorname{sn} u - k G_3 \operatorname{dn} u\right]. \tag{24b} $$

After extensive algebraic manipulation of Eqs. (24), we arrive at the desired expression

$$ \begin{aligned} u' = \frac{1}{c_1 {k_c}^2} \Bigg[& k({c_2}^2 - {c_1}^2) - G_1 \operatorname{sn} u \operatorname{dn} u + {k_c}^2 G_2 \operatorname{cn} u \operatorname{dn} u + k^3 G_3 \operatorname{sn} u \operatorname{cn} u \\ & + \left[E(u) - {k_c}^2 u\right] \left[G_1 \operatorname{cn} u + {k_c}^2 G_2 \operatorname{sn} u - k G_3 \operatorname{dn} u\right] \Bigg]. \end{aligned} \tag{25} $$

In summary, the variation of parameters solution to the perturbed Euler system of normalized Eqs. (5) for $k = c_1/c_2$ is given by Eqs. (18) with the differential equations for the parameters given by Eqs. (22) and (25). The initial integration constants are given by Eqs. (10).

Variational Equations for $c_2 < c_1$

Returning to Eq. (16) and following the same procedure but choosing $k = c_2/c_1$, we find

$$ \operatorname{sn}^{-1}(s) = c_1 (\tau - \tau_0) \triangleq u. \tag{26} $$

Using Eq. (26) along with Eqs. (9) and (BF121.00), the unperturbed solution of Euler's equations is given by

$$ \Omega_1 = c_1 \operatorname{dn} u, \tag{27a} $$
$$ \Omega_2 = c_2 \operatorname{sn} u, \tag{27b} $$
$$ \Omega_3 = c_2 \operatorname{cn} u, \tag{27c} $$
$$ u = c_1 (\tau - \tau_0). \tag{27d} $$

The variations of the two independent integrals of motion, c_1 and c_2, are obtained by direct differentiation of Eqs. (9) along with substitutions from Eqs. (5) and (27), yielding

$$ {c_1}' = \frac{1}{c_1}(\Omega_1 G_1 + \Omega_2 G_2) = G_1 \operatorname{dn} u + k G_2 \operatorname{sn} u, \tag{28a} $$
$$ {c_2}' = \frac{1}{c_2}(\Omega_2 G_2 + \Omega_3 G_3) = G_2 \operatorname{sn} u + G_3 \operatorname{cn} u. \tag{28b} $$

To develop a variational equation for u, we write the derivatives, with respect to τ, of the Eqs. (27b) and (27c) as

$$\Omega_2{}' = \frac{\partial \Omega_2}{\partial u} u' + \frac{\partial \Omega_2}{\partial c_1} c_1{}' + \frac{\partial \Omega_2}{\partial k} k', \tag{29a}$$

$$\Omega_3{}' = \frac{\partial \Omega_3}{\partial u} u' + \frac{\partial \Omega_3}{\partial c_1} c_1{}' + \frac{\partial \Omega_3}{\partial k} k'. \tag{29b}$$

Using the above to form $\Omega_2{}' \operatorname{cn} u - \Omega_3{}' \operatorname{sn} u$, we have

$$\Omega_2{}' \operatorname{cn} u - \Omega_3{}' \operatorname{sn} u = [c_2 \operatorname{dn} u]\, u' - \frac{c_2 \operatorname{dn} u}{k k_c{}^2} \left[E(u) - k_c^2 u - k^2 \operatorname{sn} u \operatorname{cn} u / \operatorname{dn} u\right] k', \tag{30a}$$

where

$$k' = \frac{1}{c_1} \left[G_3 \operatorname{cn} u + k_c{}^2 G_2 \operatorname{sn} u - k G_1 \operatorname{dn} u\right]. \tag{30b}$$

After the algebraic smoke clears from reducing Eqs. (30), we find the desired expression

$$u' = \frac{1}{c_2 k_c{}^2} \Bigg[k(c_1{}^2 - c_2{}^2) + k^3 G_1 \operatorname{sn} u \operatorname{cn} u + k_c{}^2 G_2 \operatorname{cn} u \operatorname{dn} u - G_3 \operatorname{sn} u \operatorname{dn} u + \left[E(u) - k_c{}^2 u\right] \left[G_3 \operatorname{cn} u + k_c{}^2 G_2 \operatorname{sn} u - k G_1 \operatorname{dn} u\right] \Bigg]. \tag{31}$$

In summary, the variation of parameters solution to the perturbed Euler system of normalized Eqs. (5) for $k = c_2/c_1$ is given by Eqs. (27) with the differential equations for the parameters given by Eqs. (28) and (31). The initial integration constants are given by Eqs. (10).

Variational Equations for $c_1 = c_2$

Taking $c_1 = c_2 = c$, Eq. (16) becomes

$$\int \frac{d\Omega_2}{c^2 - \Omega_2{}^2} = \int d\tau. \tag{32}$$

Choosing $s = \Omega_2/c$, we recognize

$$\int \frac{ds}{1 - s^2} = c \int d\tau \tag{33}$$

as the inverse function

$$\tanh^{-1}(s) = c\,(\tau - \tau_0) \triangleq u. \tag{34}$$

Combining this with the Jacobian elliptic function identities for $k = 1$ in Eqs. (BF122.09) and Eqs. (9), the solution of the unperturbed Eqs. (7) becomes

$$\Omega_1 = c \operatorname{sech} u, \tag{35a}$$

$$\Omega_2 = c \tanh u, \tag{35b}$$

$$\Omega_3 = c \operatorname{sech} u, \tag{35c}$$

$$u = c\,(\tau - \tau_0), \tag{35d}$$

from Eqs. (11),

$$c = \sqrt{\frac{2T}{C - A}}. \tag{35e}$$

Since $c_1 = c_2 = c$, we add Eqs. (9) together to obtain

$$\Omega_1{}^2 + 2\Omega_2{}^2 + \Omega_3{}^2 = 2c^2. \tag{36}$$

Differentiating Eq. (36) and substituting Eqs. (5) and (35) yields the variation of c

$$c' = \frac{1}{2}\left[(G_1 + G_3)\operatorname{sech} u + 2G_2 \tanh u\right]. \tag{37}$$

To develop a variational equation for u, we write the Ω_i derivatives of Eqs. (35) as

$$\Omega_2{}' = \frac{\partial \Omega_2}{\partial u} u' + \frac{\partial \Omega_2}{\partial c} c', \tag{38a}$$

$$\Omega_j{}' = \frac{\partial \Omega_j}{\partial u} u' + \frac{\partial \Omega_j}{\partial c} c' \qquad \text{for } j = 1, 3. \tag{38b}$$

Using Eqs. 38 to form $\Omega_2{}' \operatorname{sech} u - \Omega_j{}' \tanh u$, we have

$$\Omega_2{}' \operatorname{sech} u - \Omega_j{}' \tanh u = [c \operatorname{sech} u]\, u'. \tag{39a}$$

From Eqs. (35) and (5), we see that

$$\Omega_2{}' \operatorname{sech} u - \Omega_j{}' \tanh u = c^2 \left[\operatorname{sech}^3 u + \operatorname{sech} u \tanh^2 u\right] + G_2 \operatorname{sech} u - G_j \tanh u. \tag{39b}$$

Hence, the final form for the variation of u is given by

$$u' = \frac{1}{c}\left[c^2 + G_2 - G_j \sinh u\right], \tag{40}$$

where G_j can be chosen as either G_1 or G_3 without loss of generality.

In summary, the variation of parameters solution to the perturbed Euler system of normalized Eqs. (5) for $k = 1$ is given by Eqs. (35) with the differential equations for the parameters given by Eqs. (37) and (40).

Variational Equations for $c_1 = 0$ or $c_2 = 0$

Two possible circumstances can induce $k = 0$,

1. $c_1 = 0 \Rightarrow H^2 = 2TC$ or,
2. $c_2 = 0 \Rightarrow H^2 = 2TA$.

Since one constant must be zero, we call the remaining, strictly positive constant, c.

From Eq. (16), we find that for this case

$$\int \frac{d\Omega_2}{\sqrt{c^2 - {\Omega_2}^2}} = \int d\tau. \tag{41}$$

Choosing $s = \Omega_2/c$, we recognize

$$\int \frac{ds}{\sqrt{1 - s^2}} = \int d\tau \tag{42}$$

as the inverse function

$$\sin^{-1}(s) = \tau - \tau_0 \triangleq u. \tag{43}$$

Combining this with the Jacobian elliptic function identities for $k = 0$, Eqs. (BF122.08) and Eqs. (9), the solution of the unperturbed Eqs. (7) becomes

$$\Omega_1 = \begin{cases} c\cos u & \text{if } H^2 = 2TA \text{ or } c_2 = 0, \\ c & \text{if } H^2 = 2TC \text{ or } c_1 = 0. \end{cases}, \tag{44a}$$

$$\Omega_2 = c\sin u, \tag{44b}$$

$$\Omega_3 = \begin{cases} c & \text{if } H^2 = 2TC \text{ or } c_1 = 0, \\ c\cos u & \text{if } H^2 = 2TA \text{ or } c_2 = 0. \end{cases}, \tag{44c}$$

$$u = \tau - \tau_0. \tag{44d}$$

Returning again to Eqs. (11),

$$c = \begin{cases} \sqrt{\dfrac{2T}{B - A}} & \text{if } H^2 = 2TC, \\ \sqrt{\dfrac{2T}{C - B}} & \text{if } H^2 = 2TA. \end{cases} \tag{44e}$$

To find the variation of c, we begin by combining Eqs. (10) as

$$ {\Omega_1}^2 + 2{\Omega_2}^2 + {\Omega_3}^2 = c^2. \tag{45}$$

Without loss of generality, we can develop the variation of c as

$$c' = \frac{1}{c}\left[G_1\Omega_1 + 2G_2\Omega_2 + G_3\Omega_3\right], \tag{46}$$

so that with substitutions from Eqs. (44)

$$c' = \begin{cases} G_1 + 2G_2\sin u + G_3\cos u & \text{if } H^2 = 2TC, \\ G_1\cos u + 2G_2\sin u + G_3 & \text{if } H^2 = 2TA. \end{cases} \tag{47}$$

Obtaining the variation of u follows a similar procedure as in the previous section. Writing

$$\Omega_2{}' = \frac{\partial \Omega_2}{\partial u} u' + \frac{\partial \Omega_2}{\partial c} c', \tag{48a}$$

$$\Omega_j{}' = \frac{\partial \Omega_j}{\partial u} u' + \frac{\partial \Omega_j}{\partial c} c' \qquad \text{for } j = 1, 3, \tag{48b}$$

and adding $\Omega_2{}'$ and $\Omega_j{}'$ so as to eliminate c', then simplifying with substitutions from Eqs. (5) and (44), we find the variation of u is given by

$$u' = \begin{cases} \frac{1}{c}\left[c^2 + G_2 \cos u - G_3 \sin u\right] & \text{if } H^2 = 2TC, \\ \frac{1}{c}\left[c^2 + G_2 \cos u - G_1 \sin u\right] & \text{if } H^2 = 2TA. \end{cases} \tag{49}$$

In summary, the variation of parameters solution to the perturbed Euler system of normalized Eqs. (5) for $k = 1$ is given by Eqs. (44) with the differential equations for the parameters given by Eqs. (47) and (49).

Variation of $\tau_0{}'$

As previously mentioned, it is difficult to find the variation of τ_0 through the traditional applications of the variation of parameters. However, considering Eqs. (18d) and (25) for $c_1 < c_2$, the variation of τ_0 can be obtained with only a small additional effort in algebra. By directly differentiation Eq. (18d) with respect to τ, we obtain

$$u' = c_2{}' \left(\tau - \tau_0\right) + c_2 \left(1 - \tau_0{}'\right). \tag{50}$$

After substituting from Eqs. (22) and (25) into Eq. (50) and rearranging, we find an expression for the variation of τ_0

$$\tau_0' = -\frac{1}{c_1 c_2 k_c{}^2}\Bigg[\left[(E(u) - k_c{}^2 u)\,\mathrm{cn}\,u - \mathrm{sn}\,u\,\mathrm{dn}\,u\right] G_1 + \left[(E(u) - u)\,\mathrm{sn}\,u + \mathrm{cn}\,u\,\mathrm{dn}\,u\right] k_c{}^2 G_2 - \left[E(u)\,\mathrm{dn}\,u - k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\right] k G_3 \Bigg]. \tag{51}$$

This provides the variation of the originally chosen parameters: c_1, c_2, and τ_0.

This same procedure can be used to obtain a variation of τ_0 for the case of $c_2 < c_1$.

MODIFICATIONS TO AVOID SECULAR TERMS

In the variational equations for $c_1, c_2 > 0$ and $c_1 \neq c_2$, the variation of u involves secular terms as a result of the derivative of the Jacobian elliptic functions with respect to the modulus[5] k. Retention of these secular terms is not particularly desirable for any duration integration, but the effect is especially onerous for long time integrations. To remedy this, two methods to address these terms are investigated in the following.

[5] see Eqs. (BF710.51)–(BF710.53).

Brouwer Form

In the case of the $(a, e, I, \varepsilon, \widetilde{\omega}, \Omega)$ element formulation of two-body motion, the variation of the mean longitude at epoch ε involves a secular term due to the fact that ε always appears in a linear combination with a factor of the mean motion, nt. This undesirable complication arises in the expression of the derivative of the disturbing function with respect to the semi-major axis, $\partial R/\partial a$. To remove this secular term, Brouwer and Clemence[6] seeks to transform the term involving $\partial R/\partial a$ so as to move the secular contribution back into the arguments of the trigonometric functions involved as well as restructure the differential equations to isolate the large magnitude contributions. It is the latter that is presented here.

For $k = c_1/c_2$: Treating u in a similar fashion to the mean longitude λ,

$$u \equiv \rho + \eta = \int c_j \, d\tau + \eta \qquad \text{for } j = 1 \text{ or } 2 \tag{52}$$

still leaves a large number of non-secular terms in η. Rearranging Eq. (25) to match Eq. (52), the variation of u is given by

$$\begin{aligned} u' = \frac{1}{c_1 {k_c}^2} \Bigg[k({c_2}^2 - {c_1}^2) &+ \left[(E(u) - {k_c}^2 u) \operatorname{cn} u - \operatorname{sn} u \operatorname{dn} u \right] G_1 \\ &+ \left[(E(u) - {k_c}^2 u) \operatorname{sn} u + \operatorname{cn} u \operatorname{dn} u \right] {k_c}^2 G_2 \\ &- \left[(E(u) - {k_c}^2 u) \operatorname{dn} u - k^2 \operatorname{sn} u \operatorname{cn} u \right] k G_3 \Bigg], \end{aligned} \tag{53}$$

which is broken into

$$\rho' = \frac{1}{c_1 {k_c}^2} \left[k({c_2}^2 - {c_1}^2) \right], \tag{54a}$$

$$\begin{aligned} \eta' = \frac{1}{c_1 {k_c}^2} \Bigg[&\left[(E(u) - {k_c}^2 u) \operatorname{cn} u - \operatorname{sn} u \operatorname{dn} u \right] G_1 \\ &+ \left[(E(u) - {k_c}^2 u) \operatorname{sn} u + \operatorname{cn} u \operatorname{dn} u \right] {k_c}^2 G_2 \\ &- \left[(E(u) - {k_c}^2 u) \operatorname{dn} u - k^2 \operatorname{sn} u \operatorname{cn} u \right] k G_3 \Bigg], \end{aligned} \tag{54b}$$

but there are still has a significant number of non-secular terms in η'. However, if we choose

$$\begin{aligned} \rho' = \frac{1}{c_1 {k_c}^2} \Bigg[k({c_2}^2 - {c_1}^2) &+ \left[E(u) \operatorname{cn} u - \operatorname{sn} u \operatorname{dn} u \right] G_1 \\ &+ \left[E(u) \operatorname{sn} u + \operatorname{cn} u \operatorname{dn} u \right] {k_c}^2 G_2 \\ &- \left[E(u) \operatorname{dn} u - k^2 \operatorname{sn} u \operatorname{cn} u \right] k G_3 \Bigg], \end{aligned} \tag{55a}$$

[6] see Brouwer and Clemence. [3] pp. 285–286.

and

$$\eta' = -\frac{1}{c_1}\left[G_1 \operatorname{cn} u + {k_c}^2 G_2 \operatorname{sn} u - k G_3 \operatorname{dn} u\right] u, \tag{55b}$$

then, only the η portion contains any secular contribution.

For $k = c_2/c_1$: In the same fashion as the previous section, Eq. (31) becomes

$$\begin{aligned}\rho' = \frac{1}{c_2 {k_c}^2}\Bigg[k({c_1}^2 - {c_2}^2) &- \left[E(u) \operatorname{dn} u - k^2 \operatorname{sn} u \operatorname{cn} u\right] k G_1 \\ &+ \left[E(u) \operatorname{sn} u + \operatorname{cn} u \operatorname{dn} u\right] {k_c}^2 G_2 \\ &+ \left[E(u) \operatorname{cn} u - \operatorname{sn} u \operatorname{dn} u\right] G_3\Bigg],\end{aligned} \tag{56a}$$

and

$$\eta' = -\frac{1}{c_2}\left[G_3 \operatorname{cn} u + {k_c}^2 G_2 \operatorname{sn} u - k G_1 \operatorname{dn} u\right] u. \tag{56b}$$

Reciprocal Modulus Transformation

The Brouwer form of the variation of u manages to effectively isolate the secular contribution. However, it fails to move explicit secular terms into the argument of any special functions.

The reciprocal modulus transformations found in Eqs. (BF162.01) and (BF162.02) seem to indicate that it is possible to eschew the explicit secular contribution. To make the process clearer, the transforms in question are reproduced here:

$$\begin{aligned}\operatorname{sn}(ku, k_1) &= k \operatorname{sn}(u, k), \\ \operatorname{cn}(ku, k_1) &= \operatorname{cn}(u, k), \\ \operatorname{dn}(ku, k_1) &= \operatorname{dn}(u, k), \\ \operatorname{tn}(ku, k_1) &= k \operatorname{sd}(u, k),\end{aligned} \tag{BF162.01}$$

where $k_1 = 1/k$ and

$$\begin{aligned}&F(\varphi, k_1) = k F(\beta, k), \\ &K(k_1) = k\left[K(k) + i K_c(k)\right], \\ &E(\varphi, k) = \frac{1}{k}\left[E(\beta, k) - {k_c}^2 F(\beta, k)\right], \\ &\Pi(\varphi, \alpha^2, k_1) = k \Pi(\beta, \alpha^2 k^2, k),\end{aligned} \tag{BF162.02}$$

where

$$k_1 = 1/k, \qquad \beta = \sin^{-1}(k_1 \sin\varphi), \qquad k_1 \sin\varphi \le 1.$$

In the context of the previous work, several notational simplifications are made to the above. In Eqs. (BF162.01), we write

$$u_1 = ku. \tag{57}$$

Shortening the notation so that $f(u_1) \Leftrightarrow f(u_1, k_1)$ as with $\operatorname{sn} u \Leftrightarrow \operatorname{sn}(u, k)$, Eqs. (BF162.01) become

$$\operatorname{sn} u_1 = k \operatorname{sn} u, \tag{58a}$$
$$\operatorname{cn} u_1 = \operatorname{cn} u, \tag{58b}$$
$$\operatorname{dn} u_1 = \operatorname{dn} u, \tag{58c}$$
$$\operatorname{tn} u_1 = k \operatorname{sd} u. \tag{58d}$$

Next, we alter two of Eqs. (BF162.02). We recognize the first of Eqs. (BF162.02) as a combination of Eq. (57) and Eq. (58a) if we choose $\varphi = \operatorname{am}(u_1, k_1)$, $F(\varphi, k_1) = \operatorname{am}^{-1}(\varphi, k_1)$, $\beta = \operatorname{am}(u, k)$, and $F(\beta, k) = \operatorname{am}^{-1}(\beta, k)$ by (BF130.01)[7]. Then, the third of Eqs. (BF162.02) becomes

$$kE(u_1) = E(u) - {k_c}^2 u. \tag{59}$$

Finally, for a given constant modulus, Eq. (57) can be differentiated to provide a transform for the variation of u such that

$$u_1{}' = ku'. \tag{60}$$

Using the above, we proceed with the reciprocal modulus transformations.

For $k = c_1/c_2$: We start by transforming the unperturbed solution given by Eqs. (18) as

$$\Omega_1 = c_1 \operatorname{dn} u_1, \tag{61a}$$
$$\Omega_2 = c_2 \operatorname{sn} u_1, \tag{61b}$$
$$\Omega_3 = c_2 \operatorname{cn} u_1, \tag{61c}$$
$$u_1 = c_1(\tau - \tau_0), \tag{61d}$$

with $k_1 = 1/k = c_2/c_1$. Next, we transform the variation of the parameters given in Eqs. (22) and Eq. (25) so that

$$c_1{}' = G_1 \operatorname{dn} u_1 + k_1 G_2 \operatorname{sn} u_1, \tag{61e}$$
$$c_2{}' = G_2 \operatorname{sn} u_1 + G_3 \operatorname{cn} u_1, \tag{61f}$$

and

$$u_1{}' = \frac{1}{c_2 {k_{1_c}}^2} \Bigg[k_1({c_1}^2 - {c_2}^2) - \big[E(u_1) \operatorname{dn} u_1 - {k_1}^2 \operatorname{sn} u_1 \operatorname{cn} u_1\big] k_1 G_1 + \left[E(u_1) \operatorname{sn} u_1 + \operatorname{dn} u_1 \operatorname{cn} u_1\right] {k_{1_c}}^2 G_2 + \left[E(u_1) \operatorname{cn} u_1 - \operatorname{sn} u_1 \operatorname{dn} u_1\right] G_3 \Bigg], \tag{61g}$$

where k_{1_c} is the complementary reciprocal modulus given by

$${k_{1_c}}^2 = 1 - {k_1}^2. \tag{62}$$

[7] see Byrd and Friedman [4] pp. 18–19, 29.

For $k = c_2/c_1$: Again, we begin with the transform of the unperturbed solution given in Eqs. (27) so that

$$\Omega_1 = c_1 \operatorname{cn} u_1, \tag{63a}$$

$$\Omega_2 = c_1 \operatorname{sn} u_1, \tag{63b}$$

$$\Omega_3 = c_2 \operatorname{dn} u_1, \tag{63c}$$

$$u_1 = c_2(\tau - \tau_0), \tag{63d}$$

with $k_1 = 1/k = c_1/c_2$. Then, we transform the variation of the parameters given in Eqs. (28) and Eq. (31) yielding

$$c_1{}' = G_1 \operatorname{cn} u_1 + G_2 \operatorname{sn} u_1, \tag{63e}$$

$$c_2{}' = k_1 G_2 \operatorname{sn} u_1 + G_3 \operatorname{dn} u_1, \tag{63f}$$

and

$$u_1{}' = \frac{1}{c_1 {k_1}_c{}^2}\Bigg[k_1(c_2{}^2 - c_1{}^2) + [E(u_1) \operatorname{cn} u_1 - \operatorname{sn} u_1 \operatorname{dn} u_1]\, G_1$$
$$+ [E(u_1) \operatorname{sn} u_1 + \operatorname{dn} u_1 \operatorname{cn} u_1]\, {k_1}_c{}^2 G_2 - \left[E(u_1) \operatorname{dn} u_1 - k_1{}^2 \operatorname{sn} u_1 \operatorname{cn} u_1\right] k_1 G_3\Bigg], \tag{63g}$$

where ${k_1}_c$ is as specified in Eq. (62).

Summary As desired, the explicit secular terms found in Eqs. (25) and (31) vanish under the reciprocal modulus transformation; Eqs. (61g) and (63g) respectively. The price of this reduction is that the complete transformation involves all the equations which describe the motion.

While the explicit secular contribution does not appear, not all the special functions wherein the reciprocal argument retreats are periodic. In particular, the incomplete elliptic integral of the second kind is given by a pseudo-periodic relation defined as

$$E(m\pi \pm \varphi, k) = 2mE(k) \pm E(\varphi, k), \tag{BF113.02}$$

for $0 \le \varphi \le \pi/2$ and $0 \le k \le 1$.

NUMERICAL TESTS

To investigate the numerical behavior of this variation of parameters formulation, it is compared to a numerically-generated solution of the normalized Euler system found in Eqs. (5).

The problem of a constant torque applied to the body is of current interest in satellite attitude determination applications, Williams and Tanygin [14]. Table 1 shows the torque models that are used, and associates each with a tag to help identify each comparison.

The system initialization is completed by specifying values at $\tau = 0$. As with the torque models applied, Table 2 shows integration initial conditions and fixed step sizes used. These

Tag	Scaled Torques
TM00	$G_1 = G_2 = G_3 = 0$
TM01	$G_1 = G_2 = G_3 = 1$
TM02	$G_1 = G_3 = 0, G_2 = 0.25$
TM03	$G_1 = 0.004, G_2 = 0.005, G_3 = 0.003$

Table 1: Torque Models.

initial conditions were chosen to provide basic coverage of the range of the modulus since $0 < k < 1$.

Tag	Initial Conditions	Step Size
1a	$\Omega_1 = 0.5, \Omega_2 = 0, \Omega_3 = 1$	$h = 0.01$
2	$\Omega_1 = 0.999, \Omega_2 = 0, \Omega_3 = 1$	$h = 0.01$
3	$\Omega_1 = 0.2, \Omega_2 = 0, \Omega_3 = 1$	$h = 0.01$
4	$\Omega_1 = 0.5, \Omega_2 = 0, \Omega_3 = 1$	$h = 0.001$
5	$\Omega_1 = 0.5, \Omega_2 = 0, \Omega_3 = 1$	$h = 0.1$

Table 2: Initial conditions and integrator step sizes.

Using these initial conditions, the variational equations, Eqs. (22) and (25), and Euler's equations, Eqs. (5), were integrated numerically using a Runge–Kutta 4/5 scheme. Both systems of equations were integrated with the same fixed step size. The fundamental step size h was taken to be approximately $1/400^{\text{th}}$ of the initial $\operatorname{dn} u$ period as specified by the baseline initial conditions 1a, i.e. $h = K/200$, where $K \equiv K(k)$ is the complete elliptic integral of the first kind. For this case, we have[8] $K(0.5) \approx 1.686$.

Both sets of double precision (IEEE Standard 754) integrations were compared to an extended precision integration of Eqs. (5). The `doubledouble`[9] extended precision library for C++ written by Keith Briggs [2] was used in this capacity. The extended precision step size was approximately $1/10^{\text{th}}$ of the of the step size of the corresponding double precision integration. The extended precision integration was performed with a 16^{th}-degree first-order Chebyshev procedure[10].

For each integration, the L^2-norm error

$$\|\mathbf{e}\|_2 = \|\mathbf{\Omega}^* - \mathbf{\Omega}\|_2, \tag{64}$$

was computed for each step, where $\mathbf{e}$ is the difference of $\mathbf{\Omega}$ when compared with the extended precision result $\mathbf{\Omega}^*$ for $\mathbf{\Omega} = [\Omega_1 \quad \Omega_2 \quad \Omega_3]^T$. The L^2-norm of this error was graphed as a function of the time τ. Figures 1 through 19 shows the body rate error propagation results

[8] Recall that $K(0) = \pi/2$.

[9] This software library stores values using the width of two double precision numbers and manipulates them with well-known techniques due to Dekker, Linnainmaa, Kahan, Knuth and Priest to provide approximately 31 decimal digits of precision.

[10] see Richardson et al. [12].

for the torque models and initial conditions given in Tables 1 and 2. Each figure has a legend attached to identify each curve. In each legend, the curve is identified by two groups of three letter sequences where each sequence's digits represent:

1. `d` direct equation formulation,

   `v` VOP formulation using u,

   `b` VOP formulation using the Brouwer form,

   `t` VOP formulation using τ_0.

2. `d` double precision,

   `q` extended precision.

3. `c` Chebyshev ($n = 16$) integrator,

   `r` Runge-Kutta (RK45) integrator.

Lastly, much of the software used for the evaluation of the necessary special functions, i.e. complete and incomplete elliptic integrals of the first and second kinds and the Jacobian elliptic functions, was obtained from the SLATEC[11] library. Any material that was not found within SLATEC was coded from information taken from Byrd and Friedman [4].

Trends

For the figures shown, the following general trend is established: when the scaled applied torque is relatively small, each VOP formulation performs at least as well as or better than the direct equations, including when $k \approx 1$. The clear exception to this rule appears to be for the Brouwer and u VOP formulations with no applied torque, i.e. TM00. In these formulations, there is only a constant term in the related u' equation. Due to round-off in the integration procedures, the argument u accumulates "bad bits" which are presented to special functions and, as a result the true solution is tracked poorly. This appears to be less of a problem for the remaining torque models. Using the τ_0 VOP provides the expected result for TM00, providing a solution that is effectively accurate to machine precision.

The variations of step size follow the generally expected patterns. The direct integrations accuracy improves with a decrease in step size, and degrades with an increase in step size since the truncation error is a direct function of the step size. The VOP formulations remain consistent in their performance with respect to a decrease in step size. However, an increase in step size seems to improve the VOP accuracy substantially. With regard to numerical integration step size, the reader should note that care must be taken when considering step size changes since the normalized Euler equations and the variation of parameters equations each possess the same spectral content. Consequently, the variation of parameters equations cannot be integrated using step sizes that are substantially greater than that needed for the normalized equations without a substantial loss of accuracy.

[11] This software library is available from NETLIB at `http://www.netlib.org`.

Overall, it is seen that the variation of parameters solutions generally provide better accuracy than that produced by direct integration of the normalized Euler equations. This behavior was expected largely because of similar behaviors that have been observed in the numerical integration of various sets of VOP equations used in Astrodynamics applications.

CONCLUSION

We have developed a variation of parameters solution for a specific normalized form of the arbitrarily-perturbed Euler's equations. The three parameters of the variation, two amplitudes and an angular displacement or an initial time constant, lead to a very compact form for the variational equations. Additionally, the parameters c_1 and c_2 were shown to be non-negative over the regions of relevant application. This assertion permits simplified computation of the necessary special functions over the regions of interest.

The space of possible motions was categorized in terms of the modulus k. The cases of $k = 0$ and $k = 1$ marked transitions in the type of motion, and were labeled degenerate because of their effect on the resulting variational equations. The equations for these cases were presented for completeness rather than for practical use, since the motion described by the remaining two systems of equations must approach the degenerate case description as the modulus nears the boundary values.

Because the variation of the argument u contained secular contributions, two methods of addressing this complication were investigated: a Brouwer reformulation, and the reciprocal modulus transformation. In the case of the Brouwer reformulation, the variation of u was broken into two differential equations so that the secular contributions were calculated separately. Under the reciprocal modulus transformation, the obvious secular contributions are absorbed by the transform of the incomplete elliptic integral of the second kind, and the reciprocal argument u_1 appears only in the argument of special functions and possibly the torques.

In numerical tests, the accuracy of the VOP formulations were generally better than that of the direct equations, but was sensitive to the magnitude of the scaled applied torque. One disadvantage of using a VOP method is additional computational overhead. Much of this overhead is due to the calculation of the necessary special functions. This additional computation is consistent with the variation of parameters integrations of the orbital elements in Astrodynamics applications and is expected. For applications involving algebraically-complex moment components, the variational equations will produce comparable execution timings.

It should again be emphasized that the variation of parameters equations cannot be integrated using step sizes that are substantially greater than that needed for the normalized equations, since the parameter formulation contains the same spectral content as the direct equations.

ACKNOWLEDGMENT

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NOTATION

In this paper, equations taken from Byrd and Friedman [4] and from Rimrott [13] use the equation numbers from that text and are prefixed with either BF or FR, respectively. The notation $x \in \mathbb{R}$ identifies x as a real number.

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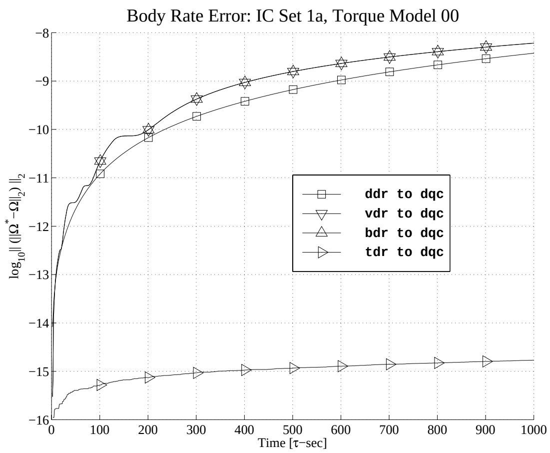

Figure 1: Body rate errors for initial conditions 1a and torque model TM00 .

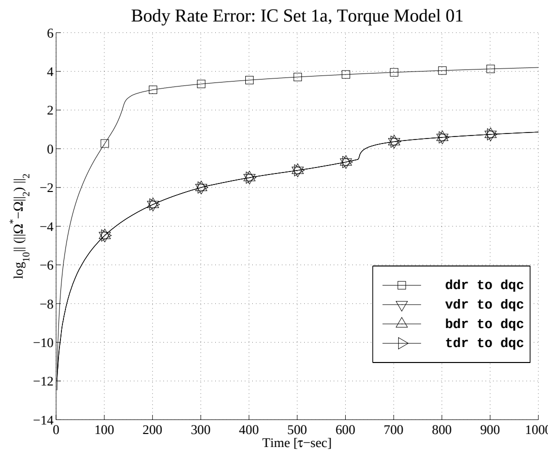

Figure 2: Body rate errors for initial conditions 1a and torque model TM01 .

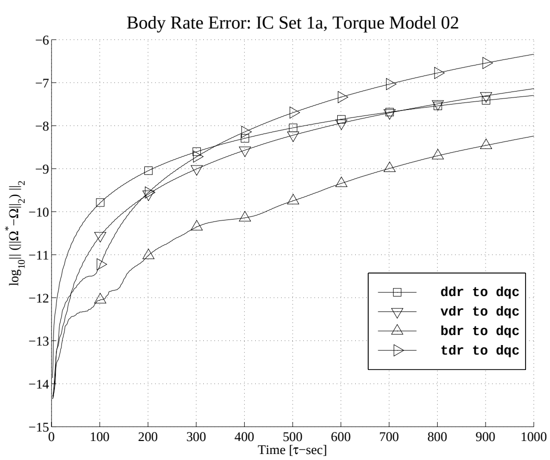

Figure 3: Body rate errors for initial conditions 1a and torque model TM02 .

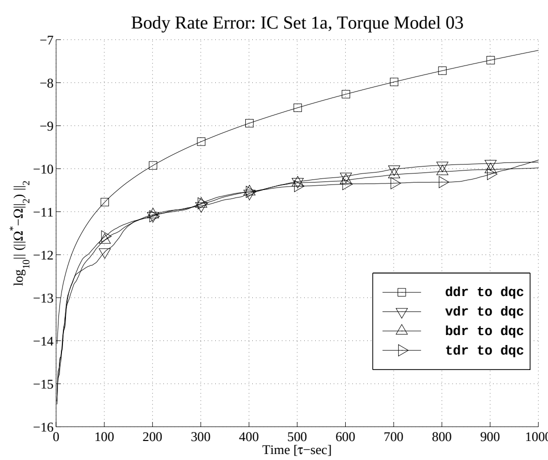

Figure 4: Body rate errors for initial conditions 1a and torque model TM03 .

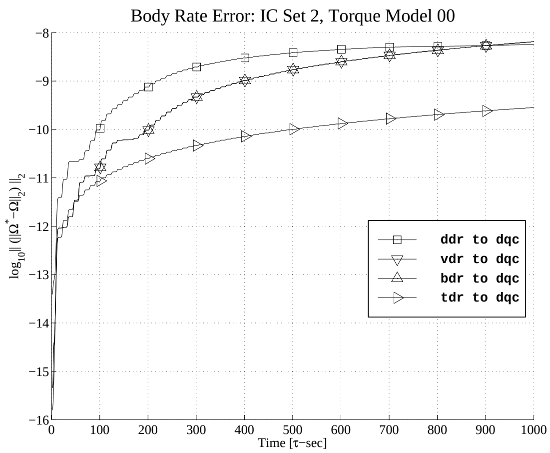

Figure 5: Body rate errors for initial conditions 2 and torque model TM00 .

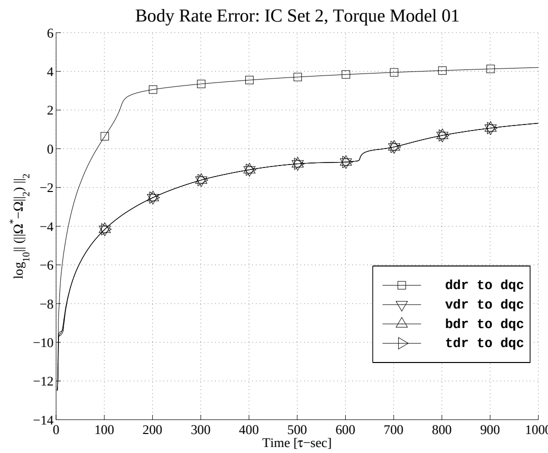

Figure 6: Body rate errors for initial conditions 2 and torque model TM01 .

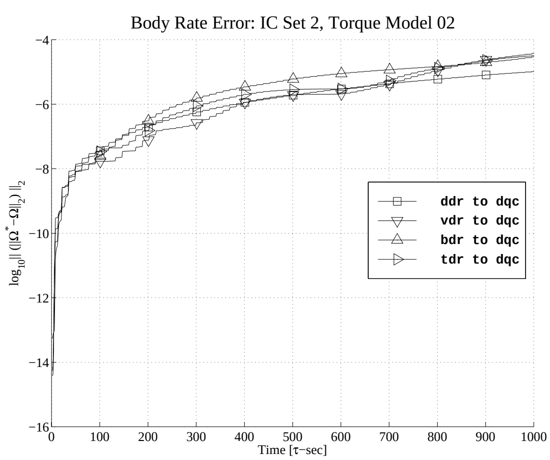

Figure 7: Body rate errors for initial conditions 2 and torque model TM02 .

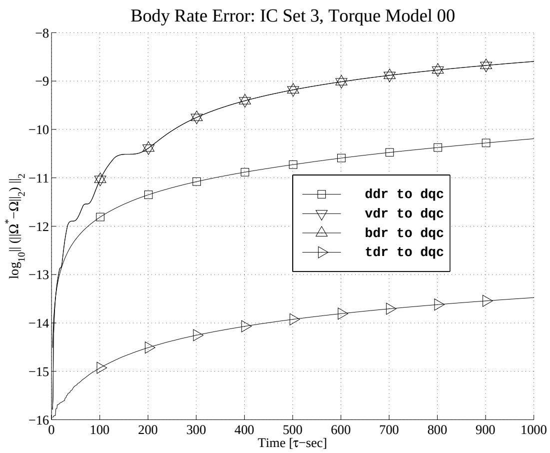

Figure 8: Body rate errors for initial conditions 3 and torque model TM00 .

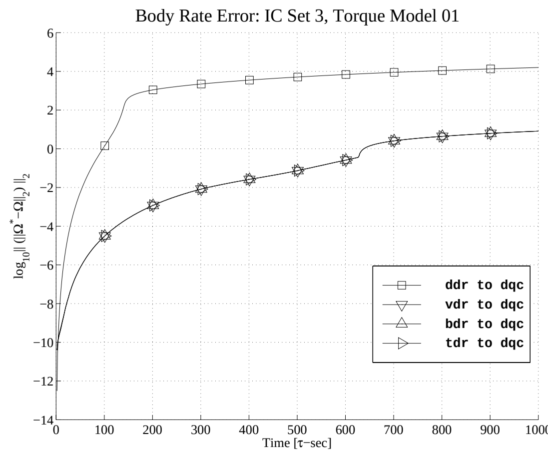

Figure 9: Body rate errors for initial conditions 3 and torque model TM01 .

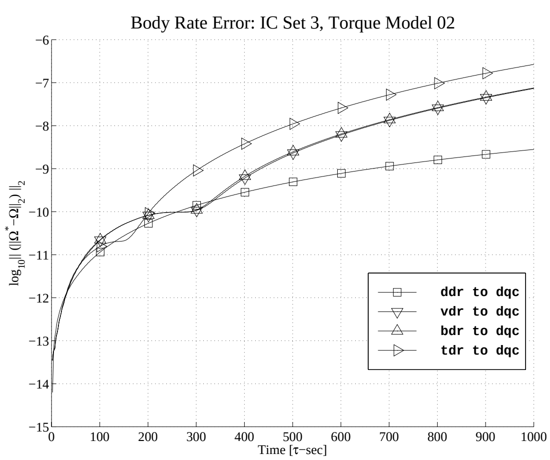

Figure 10: Body rate errors for initial conditions 3 and torque model TM02 .

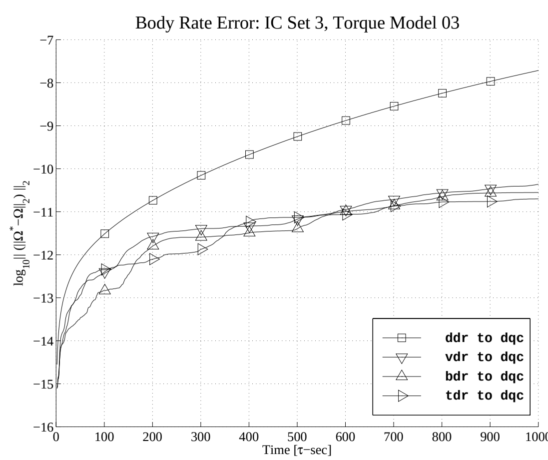

Figure 11: Body rate errors for initial conditions 3 and torque model TM03 .

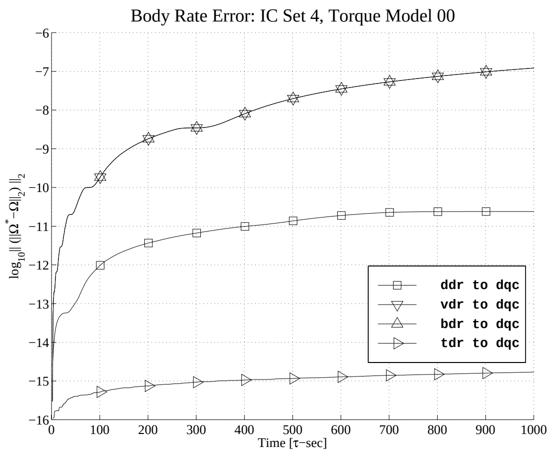

Figure 12: Body rate errors for initial conditions 4 and torque model TM00 .

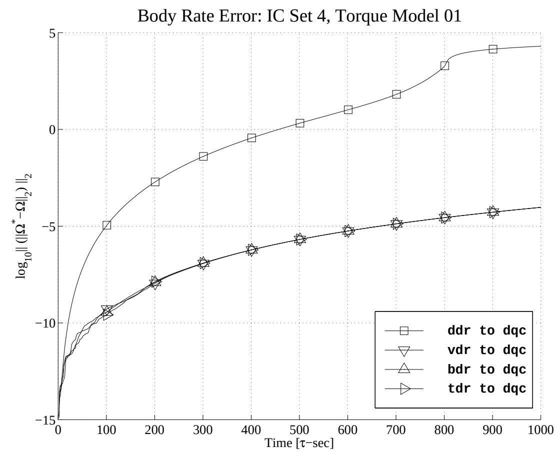

Figure 13: Body rate errors for initial conditions 4 and torque model TM01 .

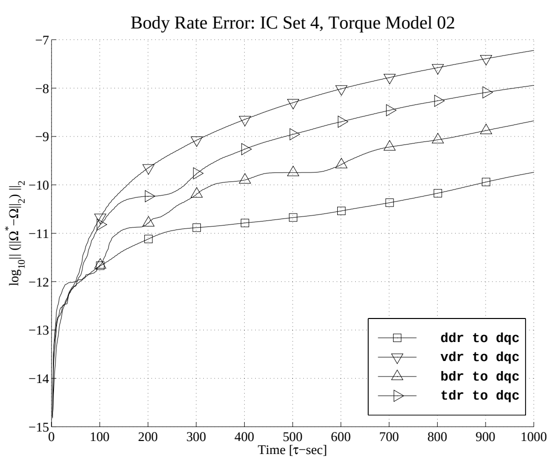

Figure 14: Body rate errors for initial conditions 4 and torque model TM02 .

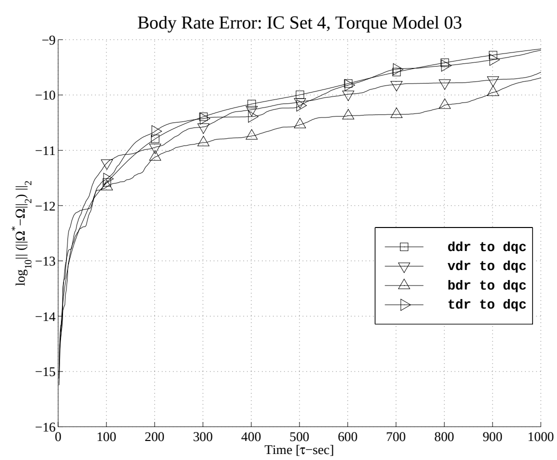

Figure 15: Body rate errors for initial conditions 4 and torque model TM03 .

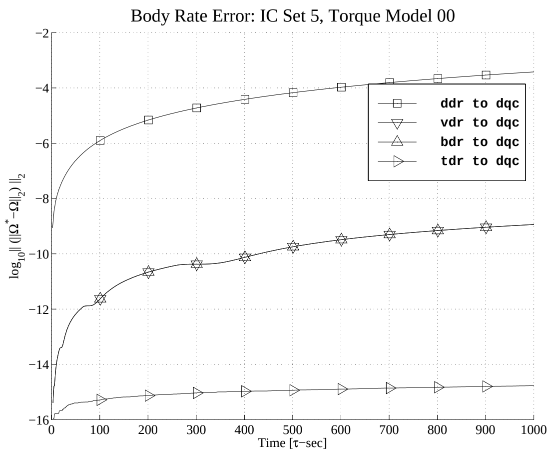

Figure 16: Body rate errors for initial conditions 5 and torque model TM00 .

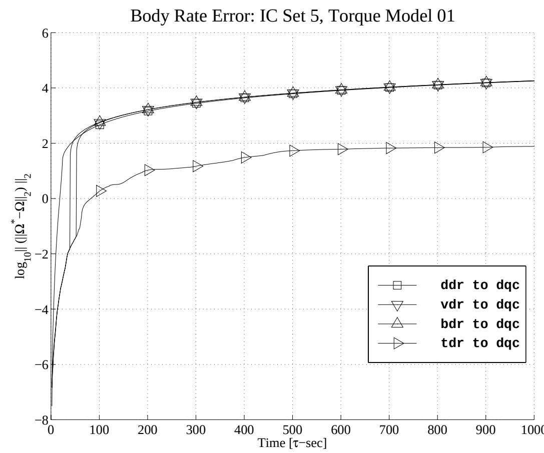

Figure 17: Body rate errors for initial conditions 5 and torque model TM01 .

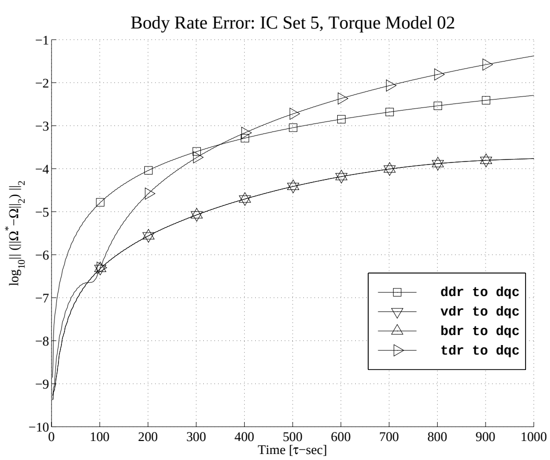

Figure 18: Body rate errors for initial conditions 5 and torque model TM02 .

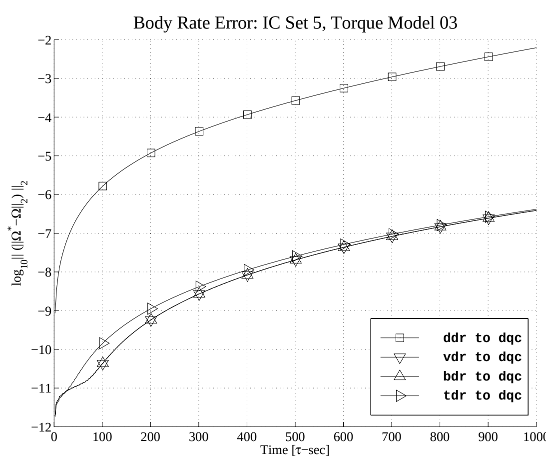

Figure 19: Body rate errors for initial conditions 5 and torque model TM03 .